

Representation Theory

A repn of a gp G on a complex vector space V is an action of G on V by linear automorphisms.

$$\text{I.e. } g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \quad \text{if } g \in G, v_1, v_2 \in V$$

$$\text{and } g \cdot (\lambda v) = \lambda g \cdot v \quad \text{if } g \in G, \lambda \in \mathbb{C}, v \in V$$

Equivalently, a representation of G on V is a gp homom

$$\rho: G \rightarrow GL(V)$$

$$GL(V) = \{L: V \rightarrow V \mid \text{linear, } L \text{ isom}\}$$

Note: We considered similar equivalences before

- perm reps: $G \leq S_{\underbrace{n}_{\text{size } n}} \iff \varphi: G \rightarrow S_n$
- G s.t. by gp action $\iff \varphi: G \rightarrow \text{Aut}(H)$

PF of equivalence

- Suppose given $\rho: G \rightarrow GL(V)$.
- Define an action of G on V as $g \cdot v = \rho(g)(v)$
- ρ homom $\implies$ this is an action, i.e.

• ρ homom $\Rightarrow$ this is an action, i.e.

$$\begin{aligned} g_1 \circ (g_2 \circ v) &= \rho(g_1)(\rho(g_2)(v)) \\ &= \rho(g_1 g_2)(v) \\ &= (g_1 g_2) \circ v \end{aligned}$$

• Action by lin. action

$$\begin{aligned} g \circ (v_1 + v_2) &= \rho(g)(v_1 + v_2) = \rho(g)(v_1) + \rho(g)(v_2) \\ &= g \circ v_1 + g \circ v_2 \end{aligned}$$

$$\begin{aligned} g \circ (\lambda v) &= \rho(g)(\lambda v) = \lambda \rho(g)(v) \\ &= \lambda g \circ v. \end{aligned}$$

Other direction: Suppose given an action $G \curvearrowright V$ by lin. action

$$\text{Define } \rho(g)(v) = g \circ v$$

For each g , have $\rho(g): V \rightarrow V$

$$\begin{aligned} \underline{\rho(g) \text{ is linear}}: \quad \rho(g)(v_1 + v_2) &= g \circ (v_1 + v_2) = g \circ v_1 + g \circ v_2 \\ &= \rho(g)(v_1) + \rho(g)(v_2) \end{aligned}$$

$$\rho(g)(\lambda v) = g \circ (\lambda v) = \lambda g \circ v = \lambda \rho(g)(v)$$

Because $G \curvearrowright V$ is an action,

— $\rho(g)$ is an isomorphism

— Because $G \curvearrowright V$ is an action, $\rho(g)$ is a homom. 

Example

Example

- ① G arbitrary, $V = \mathbb{C}$. $\rho: G \rightarrow GL(V)$ as
 $\rho(g) = 1 \quad \forall g \in G.$

In other words, $g \cdot v = v \quad \forall v \in V, g \in G.$

Called the trivial repn.

- ② $G = S_n$, $V = \mathbb{C}^n$ with action

$$\rho \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = L(\rho) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

$$L: S_n \rightarrow GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$$

I.e. $\rho: S_n \rightarrow GL(V)$ is given by $\rho(p) = L(p)$

Equiv.
$$\rho \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_{p^{-1}(1)} \\ a_{p^{-1}(2)} \\ \vdots \\ a_{p^{-1}(n)} \end{pmatrix}$$

- ② $V = \mathbb{C}$, $G = S_n$, $\text{sgn}: S_n \rightarrow \pm 1$

Define $g \cdot v = \text{sgn}(g) v$

- ③ Notation as in example 1. Define $W \subseteq V = \mathbb{C}^n$
as $W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1 + a_2 + \dots + a_n = 0 \right\}.$

$$\text{an } W = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} : a_1 + a_2 + \dots + a_n = 0 \right\}.$$

$$S_n \text{ acts on } W \text{ by } p \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = L(p) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_{p^{-1}(1)} \\ \vdots \\ a_{p^{-1}(n)} \end{pmatrix}.$$

If $w \in W$, $p \in S_n$, then $p \cdot w = L(p)w \in W$.

This is a rep'n of S_n .

④ $G = \mathbb{Z}/n\mathbb{Z}$, $V = \mathbb{C}$, then $\text{GL}(V) = \mathbb{C}^\times$

Define $\rho: G \rightarrow \mathbb{C}^\times = \text{GL}(V)$ as

$$\rho(k) = e^{2\pi i k/n} \in \mathbb{C}^\times.$$

Note: ① This is well-defined: if $k \equiv k' \pmod{n}$, then

$$e^{2\pi i k/n} = e^{2\pi i k'/n}$$

$$\begin{aligned} k' = k + nm &\Rightarrow e^{2\pi i k'/n} = e^{2\pi i (k+nm)/n} \\ &= e^{2\pi i k/n} \underbrace{e^{2\pi i m}}_{=1} = e^{2\pi i k/n} \end{aligned}$$

② gp homom: $\rho(k_1 + k_2) = e^{2\pi i (k_1 + k_2)/n}$

$$\begin{aligned} &= e^{2\pi i k_1/n} e^{2\pi i k_2/n} \\ &= \rho(k_1) \rho(k_2) \end{aligned}$$

(4') $l \in \mathbb{Z}$. $G = \mathbb{Z}/n\mathbb{Z}$, $V = \mathbb{C}$,
 then $k \in G$, $k \cdot v = e^{2\pi i k l / n} \cdot v$

(5) $G = D_n$, $V = \mathbb{C}^2$. Define

$$\rho: D_n \rightarrow GL(V) = GL_2(\mathbb{C})$$

$$D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

$$\text{as } \rho(x) = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix},$$

$$\rho(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall: We checked ρ is a homom.

(5') $G = D_n$, $V = \mathbb{C}^2$, $D_n = \langle x, y \mid x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$

$$\rho': G \rightarrow GL_2(\mathbb{C}) \text{ as}$$

$$\rho'(x) = \begin{pmatrix} e^{2\pi i / n} & 0 \\ 0 & e^{-2\pi i / n} \end{pmatrix}$$

$$\rho'(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Note: $\cdot \rho'(x^n) = \rho'(x)^n = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}^n = 1$

$\cdot \rho'(y^2) = \rho'(y)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1$

$\cdot \rho'(y)\rho'(x)\rho'(y)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
 $= \begin{pmatrix} e^{-2\pi i/n} & \\ & e^{2\pi i/n} \end{pmatrix} = \rho'(x)^{-1}$

Examples (5) and (5') are said to be equivalent, or isomorphic

$\cdot$ Define $b_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$\cdot b_1, b_2$ is another basis of $\mathbb{C}^2$, $\underbrace{\begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\text{usual basis}}$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{i\theta} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{-i\theta} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Next time: Make precise change of basis/equiv reps.
